Tutorial 1 : Selected problems of Assignment 1

Leon Li

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Contact Information

Name : Li Yan Lung, Leon

Email : ylli@math.cuhk.edu.hk

Office : Academic Building 1, Room 505

Office hour : Tue, Wed, Thurs : 2:00 p.m. - 5:00 p.m.

Assumption Throughout the tutorial, $f : [-\pi, \pi] \to \mathbb{R}$ is a 2π-periodic integrable function with Fourier series

$$S(f) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Q1) (Ex. 1, Q3) Show that if f is an even function, then $S(f)$ is a cosine series, i.e. $b_n = 0$, $\forall n \geq 1$

Sol Recall that $\pi b_n = \int_{-\pi}^{\pi} f(x) \sin nx \, dx = 0$

since $f(x) \sin nx$ is an odd function.

Details: $\int_{-\pi}^{\pi} f(x) \sin nx \, dx = \left(\int_0^{\pi} + \int_{-\pi}^{0}\right)\left(f(x) \sin nx \, dx\right) = \mathrm{I} + \mathrm{II}$

For II, apply a change of variables $x = -y$; $dx = -dy$

$\therefore \int_{-\pi}^{0} f(x) \sin nx \, dx = \int_{\pi}^{0} f(-y) \sin(-ny)(-dy) = -\int_0^{\pi} f(y) \sin ny \, dy = -\mathrm{I}$

$\therefore \pi b_n = \mathrm{I} + \mathrm{II} = 0$. Hence $b_n = 0$

Q2) (Ex. 1, Q6) Suppose in addition, f is differentiable such that $f' : [-\pi, \pi] \to \mathbb{R}$ is integrable

Show that Riemann - Lebesgue lemma holds for such f:

$$\lim_{n\to\infty} |a_n| = 0 = \lim_{n\to\infty} |b_n|$$

Sol Showing $\lim_n |b_n| = 0$: Note that $\forall n \geq 1$,

$$\int_{-\pi}^{\pi} f'(x) \cos nx \, dx$$

$$= [f(x) \cos nx]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} f(x)(-n \sin nx)\, dx$$

$$= 0 + n\pi b_n = n\pi b_n$$

Also, since f' is bounded. $\exists M > 0$ s.t. $\|f'\|_\infty \leq M$

$$\therefore |b_n| = \frac{1}{\pi n} \left| \int_{-\pi}^{\pi} f'(x) \cos nx \, dx \right| \leq \frac{1}{\pi n} \int_{-\pi}^{\pi} M \, dx = \frac{2M}{n}$$

$$\therefore \lim_{n\to\infty} |b_n| = 0$$

Similarly for a_n: $\int_{-\pi}^{\pi} f'(x) \sin nx \, dx = -n\pi a_n$, $\forall n \geq 1$

$\therefore |a_n| = \frac{1}{\pi n} \left| \int_{-\pi}^{\pi} f'(x) \cos nx \, dx \right| \leq \frac{2M}{n}$. Therefore, $\lim_{n\to\infty} |a_n| = 0$

Q3) (Ex. 1, Q7) Using Q2, Show that Riemann-Lebesgue lemma holds for any f in Assumption : $\lim_{n\to\infty} |a_n| = 0 = \lim_{n\to\infty} |b_n|$

Sol Showing $\lim_{n\to\infty} |b_n| = 0$: $\forall \varepsilon > 0$, $K \in \mathbb{N}$ to be determined

Lemma (a) There exists a step function $S : [-\pi, \pi] \to \mathbb{R}$, i.e.

$$S(x) = \begin{cases} m_1, & x \in [\overset{x_0}{\overset{\parallel}{-\pi}}, x_1] \\ m_2, & x \in (x_1, x_2] \\ \vdots & \\ m_N, & x \in (x_{N-1}, \underset{\underset{x_N}{\parallel}}{\pi}] \end{cases}$$

such that $0 < \int_{-\pi}^{\pi} (f - S) < \frac{\varepsilon}{3}$

(b) For each step function $S : [-\pi, \pi] \to \mathbb{R}$, there exists a C^1 function $g : [-\pi, \pi] \to \mathbb{R}$ such that $\int_{-\pi}^{\pi} |S - g| < \frac{\varepsilon}{3}$

Assuming the Lemmas : by Q2, $\exists K \in \mathbb{N}$ s.t. $\forall n \geq K$

$$\left| \int_{-\pi}^{\pi} g(x) \sin nx \, dx \right| < \frac{\varepsilon}{3}$$

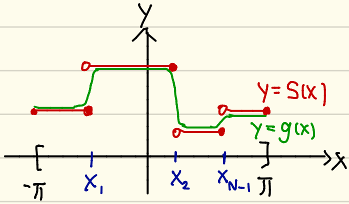
$$\therefore \pi |b_n| = \left| \int_{-\pi}^{\pi} f(x) \sin nx \, dx \right| = \left| \int_{-\pi}^{\pi} [(f - S) + (S - g) + g] \sin nx \, dx \right|$$

$$\leq \int_{-\pi}^{\pi} (f - S) + \int_{-\pi}^{\pi} |S - g| + \left| \int_{-\pi}^{\pi} g(x) \sin nx \, dx \right| \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \ \forall n \geq K$$

$\therefore \lim_{n\to\infty} |b_n| = 0$. Similarly, $\lim_{n\to\infty} |a_n| = 0$ by replacing $\sin nx$ by $\cos nx$ above.

Proof of Lemma (a) See Lecture note 1, Lemma 1.3

(b) Pictorial proof:



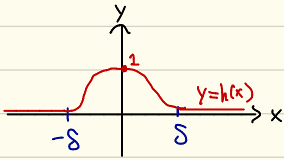
Analytic details:

Step 1: $\forall \delta > 0$, define $h: \mathbb{R} \longrightarrow \mathbb{R}$ by $h(x) = \begin{cases} e^{-\frac{1}{\delta^2 - x^2}}, & |x| < \delta \\ 0, & |x| \geq \delta \end{cases}$

bump function

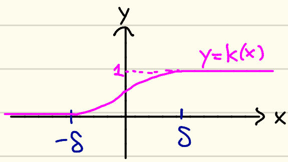
(Exercise) h is smooth (i.e. all derivatives exist everywhere)

with $\begin{cases} h(x) \geq 0, \ \forall x \in \mathbb{R} \\ \int_{\mathbb{R}} h(x)\,dx = A < +\infty \end{cases}$



Step 2: $\forall \delta > 0$, define $k: \mathbb{R} \longrightarrow \mathbb{R}$ by $k(x) = \frac{1}{A} \int_{-\infty}^{x} h(t)\,dt$

(Exercise) k is smooth

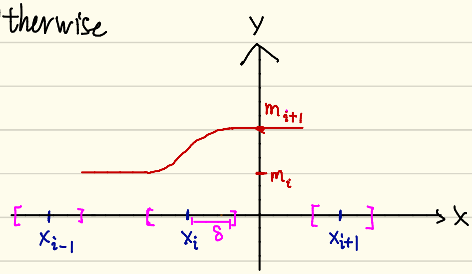


with $\begin{cases} k(x) = 0, \ \forall x \leq -\delta \\ 0 < k(x) < 1, \ \forall -\delta < x < \delta \\ k(x) = 1, \ \forall x \geq \delta \end{cases}$

Step 3: Given $s(x)$, choose $0<\delta<\min\limits_{1\le i\le N}\{\frac{1}{2}(x_i-x_{i-1})\}$, define

$$g(x)=\begin{cases}(m_{i+1}-m_i)\,k(x-x_i)+m_i\,, & \forall\, x\in[-\pi,\pi]\cap\overline{B}_\delta(x_i)\\ s(x) & \text{Otherwise}\end{cases}$$

Picture near x_i :



Then g is C^1 (in fact is smooth)

and $\int_{-\pi}^{\pi}|s-g| = \sum\limits_{i=1}^{N}\int_{[-\pi,\pi]\cap B_\delta(x_i)}|s-g| \le \sum\limits_{i=1}^{N} 2\delta\cdot\left(2\max\{|m_i|,|m_{i+1}|\}\right) < \frac{\varepsilon}{3}$

by choosing $0<\delta<\frac{\varepsilon}{3}\cdot\frac{1}{4Nm+1}$, $m:=\max\limits_{0\le i\le N}\{|m_i|\}$